JOURNAL OF APPROXIMATION THEORY 47, 270–283 (1986)

Derived and Integrated Sets of Simple Sets of Polynomials in Two Complex Variables

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Communicated by Oved Shisha

Received November 3, 1980; revised August 15, 1984

In the present paper, we illustrate the contrast, concerning the effectiveness properties of the derived sets of simple sets of polynomials, between the single- and the two-variable cases. Moreover, a positive result is established for the relationship between the Cannon functions of simple sets of polynomials in two complex variables and those of the derived sets. Finally, it is shown that concerning the effectiveness of the integrated sets, the result of the single-variable case can be extended, without undue difficulty, to two variables.

1. NOTATION AND PRELIMINARIES

The derived and integrated sets of a given basic set of polynomials of a single complex variable have been studied by many authors, of whom we may mention Makar [1, Theorems I–IV] and Newns [3, Theorems 23.1, 23.2]. According to the results obtained by these authors, and noting that the outstanding restrictions of Makar [1, formulas (13), p. 220; (14), p. 222) apply for simple sets, it can be stated that *both the derived and integrated sets of a given simple set of polynomials retain the effectiveness properties possessed by the given set.* We propose to investigate the extent of generalisation of the above statement for the two-variable case. The present work was motivated by the fact, illustrated here by the example of Section 2 below, that, in contrast with the single-variable case, the effectiveness properties of the derived set of a given simple set of polynomials in two complex variables may be distinct from those of the given set.

The main results obtained in the present paper are displayed in

0021-9045/86 $3.00

Theorems 1 and 2 below and to formulate them we must first develop certain preliminaries and notation.

The reader is assumed to be acquainted with the theory of basic sets of polynomials in a single complex variable as given by Whittaker [4, 5] and of polynomials in several complex variables as given by [2]. We first give a detailed account of simple sets of polynomials in two complex variables. Thus, in the space $\mathbb{C}^2$ of the two complex variables z and w, the successive monomials $1, z, w; z^2, zw, w^2;$... are arranged so that the enumeration number of the monomial $z^j w^k$ in the above sequence is $\frac{1}{2}(j+k)(j+k+1)+k$ $(j, k \geqslant 0)$. The enumeration number of the last monomial of a polynomial $p(z, w)$ in two complex variables is called the *degree of the polynomial.* A sequence $\{p_i(z; w\}_0^\infty$ of polynomials in two complex variables in which the order of each polynomial is equal to its degree is called a *simple set.* Such a set is conveniently denoted $\{p_{m,n}(z; w)\}$, where the last monomial in $p_{m,n}(z, w)$ is $z^m w^n$. If further, the coefficient of this last monomial is 1, the simple set is termed *monic.* Thus, in the simple monic set $\{p_{m,n}(z; w)\}$ the polynomial $p_{m,n}(z, w)$ is represented as follows.

$$p_{m,n}(z, w) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} p_{k-j,j}^{m,n} z^{k-j} w^j \qquad (p_{m,n}^{m,n} = 1;\ p_{m+n-j,j}^{m,n} = 0,\ j > n). \quad (1.1)$$

The fact that the simple set $\{p_{m,n}(z, w)\}$ is necessarily basic follows from the observation that the matrix $[p_{h,i}^{m,n}]$ of coefficients of the polynomials of the set is a lower triangular matrix with non-zero diagonal elements. (These elements are each equal to 1 for monic sets.) In this matrix the coefficients $(p_{h,i}^{m,n})$ are lexicographically arranged in rows with respect to the subscripts (h, i) and in columns with respect to the superscripts (m, n). This lower triangular matrix has an inverse, also a lower triangular matrix $[\bar{p}_{h,i}^{m,n}]$, in terms of which the following representation holds:

$$z^m w^n = \sum_{k=0}^{m+n} \sum_{j=0}^{k} \bar{p}_{k-j,j}^{m,n} p_{k-j,j}(z; w) \qquad (\bar{p}_{m,n}^{m,n} = 1;\ \bar{p}_{m+n-j,j}^{m,n} = 0,\ j > n). \quad (1.2)$$

To investigate the effectiveness properties of the set $\{p_{m,n}(z; w)\}$ we first form the *Cannon sum of the set*, defined as

$$\omega_{m,n}[r] = \sigma_{m,n} \sum_{k=0}^{m+n} \sum_{j=0}^{k} |\bar{p}_{k-j,j}^{m,n}|\, M[p_{k-j,j}; r]. \quad (1.3)$$

where

$$M[p_{h,i}; r] = \sup |p_{h,i}(z; w)| \quad (1.4)$$

over the closed sphere $\bar{S}_r$ (cf. [2, formula (2.4), p. 44]),

$$\sigma_{m,n} = \frac{(m+n)^{(m+n)/2}}{m^{m/2} n^{n/2}} \qquad (m, n > 0);$$
$$= 1 \qquad (m, n = 0). \tag{1.5}$$

The quantity (1.5) will be used often in the subsequent work. Also, the notation (1.4) will be adopted here for functions regular in the sphere $\bar{S}_r$. The effectiveness properties of the set $\{p_{m,n}(z; w)\}$ are governed by the *Cannon function of the set*, given by

$$\omega[r] = \limsup_{m+n \to \infty} \{\omega_{m,n}[r]\}^{1/(m+n)} \geqslant r. \tag{1.6}$$

As a typical effectiveness result, we may mention here Cannon's theorem for two variables, in the following form (cf. [2, Theorem 3]).

CANNON'S THEOREM. *A necessary and sufficient condition for the simple set $\{p_{m,n}(z; w)\}$ to be effective in $\bar{S}_r$ is that*

$$\omega[r] = r. \tag{1.7}$$

By effectiveness of the set in $\bar{S}_r$ we mean that the set forms a base for the class E of functions regular in $\bar{S}_r$ with a norm given by $M[f; r]$ for each $f \in E$.

We now define the *derived set with respect to* z, namely, *the derived set of the simple monic set* $\{p_{m,n}(z; w)\}$ to be the set $\{u_{m,n}(z; w)\}$ given by

$$u_{m,n}(z, w) = \frac{\partial}{\partial z} p_{m+1,n}(z; w) \qquad (m, n \geqslant 0). \tag{1.8}$$

Since the last term in the polynomial $u_{m,n}(z; w)$ is $\partial z^{m+1} w^n / \partial z = (m+1) z^m w^n$ it follows that the derived set $\{u_{m,n}(z; w)\}$ is simple (but not monic).

Needless to say, an identical procedure can be carried out for the treatment of the derived sets with respect to w.

The following notation is introduced. Write

$$t_{n,h} = \bar{p}_{0,h+1}^{0,n+1}; \qquad 0 \leqslant h \leqslant n; \; t_{n,h} = 0, \; h > n, \tag{1.9}$$

and put

$$t_n(z) = \sum_{h=0}^{n} t_{n,h} z^h.$$

Since the set $\{p_{m,n}(z; w)\}$ is monic then $\bar{p}^{0,n+1}_{0,n+1} = 1$ and therefore, the set $\{t_n(z)\}$ is a simple monic set of polynomials of the single variable z. Suppose that z^n admits the representation

$$z^n = \sum_{h=0}^{n} \bar{t}_{n,h} t_h(z). \tag{1.10}$$

Then, as usual, the Cannon sum of the set $\{t_n(z)\}$ will be

$$\lambda_n(r) = \sum_{h=0}^{n} |\bar{t}_{n,h}| \, M(t_h; r), \tag{1.11}$$

where $M(t_h; r) = \sup |t_h(z)|$ in $|z| \leqslant r$. The Cannon function of the same set is

$$\lambda(r) = \limsup_{n \to \infty} \{\lambda_n(r)\}^{1/n}. \tag{1.12}$$

The first main result of the present work establishes a relationship between the Cannon functions of the given simple monic set of polynomials and the derived set. This result, which is formulated in Theorem 1 below, generalises, to the two-variable case, the main inequality of Makar [1, formula (11), p. 220], concerning the same items in a single variable. With the above notation, Theorem 1 can be stated as follows.

THEOREM 1. *For any positive number r, the Cannon function* $\Omega\,[r]$ *of the derived set* $\{u_{m,n}(z; w)\}$, *of the simple monic set* $\{p_{m,n}(z; w)\}$, *will satisfy the inequality*

$$\Omega[r] \leqslant \lambda(r) \{\omega[r]/r\}^2, \tag{1.13}$$

and this inequality cannot be improved when the set $\{p_{m,n}(z; w)\}$ *is effective in* $\bar{S}_r$.

We then define the *integrated set with respect to z*, namely, the *integrated set of the given simple* (not necessarily monic) *set* $\{p_{m,n}(z; w)\}$ to be the set $\{v_{m,n}(z; w)\}$, constructed in the following manner.

For $m > 0$, we have

$$v_{m,n}(z; w) = \int_0^z p_{m-1,n}(\zeta; w)\, d\zeta \qquad (n \geqslant 0), \tag{1.14}$$

and when $m = 0$, we set

$$v_{0,n}(z; w) = w^n \qquad (n \geqslant 0). \tag{1.15}$$

Since the last term in $v_{m,n}(z, w)$ is $(1/m)\, p_{m-1,n}^{m-1,n}\, z^m w^n$, $m > 0$, it follows that the integrated set $\{v_{m,n}(z; w)\}$ is a simple set (but not necessarily monic). Again, an identical treatment can be carried out for the integrated sets with respect to w.

With the foregoing choice of integrated sets, we see from the following theorem, which is the second main result of the present paper, that the effectiveness properties of the integrated set $\{v_{m,n}(z; w)\}$ are, as in the single-variable case (cf. Makar [1, Theorem IV]), identical with those of the given simple set $\{p_{m,n}(z; w)\}$.

THEOREM 2. *Let $\{p_{m,n}(z; w)\}$ be a given simple set of polynomials and suppose that $\{v_{m,n}(z; w)\}$ is the integrated set defined by* (1.14) *and* (1.15). *Then the Cannon functions of the sets $\{p_{m,n}(z; w)\}$ and $\{v_{m,n}(z; w)\}$, corresponding to any positive value of r, are equal provided that they are finite.*

2. EXAMPLE

We construct, in what follows, a simple monic set of polynomials in two complex variables, such that the effectiveness properties of the derived set are distinct from those of the constructed set.

In fact, consider the set $\{p_{m,n}(z; w)\}$, constructed as follows.

$$\begin{aligned} &p_{0,0}(z; w) = 1; \qquad p_{m,n}(z; w) = z^m w^n && (m, n > 0),\\ &p_{m,0}(z; w) = z^m + \sum_{j=0}^{m-1} f(j)(z^j + z^{j-1}w + \cdots + w^j) && (m \geqslant 1),\\ &p_{0,n}(z; w) = \frac{1}{f(n)} \sum_{j=0}^{n} f(j)(z^j + z^{j-1}w + \cdots + w^j) && (n \geqslant 1), \end{aligned} \tag{2.1}$$

where

$$f(j) = 1 + \left(\frac{j+3}{j+1}\right)^{(j+1)/2} \qquad (j \geqslant 0). \tag{2.2}$$

It is seen that this set is simple and monic. Moreover, the following representations are obvious.

$$\begin{aligned} z^m w^n &= p_{m,n}(z; w) && (m, n > 0),\\ z^m &= p_{m,0}(z; w) - f(m-1)\, p_{0,m-1}(z; w) && (m \geqslant 1). \end{aligned} \tag{2.3}$$

Also, it can be verified, for $n \geqslant 1$, that

$$w^n = p_{0,n}(z; w) - \sum_{j=0}^{n-1} p_{n-j,j}(z; w) + f(n-1)\left\{1 - \frac{1}{f(n)}\right\} p_{0,n-1}(z; w). \tag{2.4}$$

In fact, the formula (2.4) is obvious for $n = 1$. When $n > 1$, the right-hand sider of (2.4), in view of (2.1), will be equal to

$$\begin{aligned} z^n + w^n + \sum_{j=1}^{n-1} z^{n-j}w^j + \frac{1}{f(n)} \sum_{j=0}^{n-1} f(j)(z^j + z^{j-1}w + \cdots + w^j) \\ - z^n - \sum_{j=0}^{n-1} f(j)(z^j + z^{j-1}w + \cdots + w^j) - \sum_{j=1}^{n-1} z^{n-j}w^j \\ + \left\{1 - \frac{1}{f(n)}\right\} \sum_{j=0}^{n-1} f(j)(z^j + z^{j-1}w + \cdots + w^j) = w^n, \end{aligned}$$

and (2.4) is thus verified for $n \geqslant 1$.

The Cannon sum of the set $\{p_{m,n}(z; w)\}$, as defined in (1.3), can be evaluated rom the representations (2.3) and (2.4), taking (2.1) and (2.2) into account. Thus, we shall have

$$\begin{aligned} &\omega_{m,n}[r] = r^{m+n} && (m, n, r > 0), \\ &\omega_{m,0}[r] < \{1 + 2em(m+1)\} r^m && (m, r \geqslant 1), \\ &\omega_{0,n}[r] < \{2n + 1 + 2en(n+1)\} r^n && (n, r \geqslant 1). \end{aligned}$$

Therefore, for the Cannon function of the set $\{p_{m,n}(z; w)\}$, as given by (1.6), we shall have

$$\omega[r] \leqslant r \qquad (r \geqslant 1).$$

Hence, by Cannon's theorem, we deduce that the set $\{p_{m,n}(z; w)\}$ will be effective in all $\bar{S}_r$ for $r \geqslant 1$.

According to (1.8), the derived set $\{u_{m,n}(z; w)\}$ of the set $\{p_{m,n}(z; w)\}$ of (2.1), is given as follows.

$$\begin{aligned} &u_{0,0}(z; w) = 1; \qquad u_{m,n}(z; w) = (m+1)\, z^m w^n \qquad (m \geqslant 0; n \geqslant 1), \\ &u_{m,0}(z; w) = (m+1)z^m + \sum_{j=0}^{m-1} f(j+1)\{(j+1)z^j + jz^{j-1}w + \cdots + w^j\} \qquad (m \geqslant 1). \end{aligned} \tag{2.5}$$

As for the monomials ($z^m w^n$) representation in terms of the polynomials $\{u_{m,n}(z; w)\}$, the following formula will be established for $m \geqslant 2$.

$$(m+1)z^m = u_{m,0}(z; w) - f(m) \sum_{j=0}^{m-1} u_{m-j-1,j}(z; w) + \sum_{k=0}^{m-2} (-1)^{m-k} f(k+1) \left[\prod_{j=k+2}^{m} \{f(j)-1\} \right] \sum_{j=0}^{k} u_{k-j,j}(z; w). \tag{2.6}$$

Actually, it can be verified from (2.5) that the formula (2.6) is satisfied for $m = 2$. Suppose that (2.6) is valid for certain $m \geqslant 2$; then according to (2.5) we shall have

$$u_{m+1,0}(z; w) = (m+2)z^{m+1} + \{f(m+1)-1\} (m+1) z^m + u_{m,0}(z; w) + f(m+1) \sum_{j=1}^{m} u_{m-j,j}(z; w).$$

Therefore, applying (2.6), we can easily obtain

$$(m+2) z^{m+1} = u_{m+1,0}(z; w) - f(m+1) \sum_{j=0}^{m} u_{m-j,j}(z; w) + \sum_{k=0}^{m-1} (-1)^{m+1-k} f(k+1) \times \left[\prod_{j=k+2}^{m+1} \{f(j)-1\} \right] \sum_{j=0}^{k} u_{k-j,j}(z; w),$$

so that (2.6) remains valid for $m+1$ also. Hence, by induction we infer that (2.6) is true for all $m \geqslant 2$.

We shall denote by $\Omega_{m,n}[r]$ for the Cannon sum of the derived set $\{u_{m,n}(z; w)\}$. Therefore, as in (1.3), we can deduce from (2.6) that

$$\Omega_{m,0}[r] > \frac{f(1)}{m+1} \left[\prod_{j=2}^{m} \{f(j)-1\} \right] M[u_{0,0}; r] \qquad (m \geqslant 2). \tag{2.7}$$

Introducing therefore (2.2) in (2.7, it follows that

$$\Omega_{m,0}[r] > \frac{1}{m+1} \prod_{j=1}^{m} \left(\frac{j+3}{j+1} \right)^{(j+1)/2}.$$

Hence, the Cannon function of the set $\{u_{m,n}(z;w)\}$ is

$$\begin{aligned}
\Omega[r] &= \limsup_{m+n\to\infty} \{\Omega_{m,n}[r]\}^{1/(m+n)} \\
&\geqslant \lim_{m\to\infty} \{\Omega_{m,0}[r]\}^{1/m} \\
&\geqslant \lim_{m\to\infty} \left\{\frac{1}{m+1}\prod_{j=1}^{m}\left(\frac{j+3}{j+1}\right)^{(j+1)/2}\right\}^{1/m} \\
&= \lim_{m\to\infty}\left(\frac{m+3}{m+1}\right)^{(m+1)/2} = e.
\end{aligned} \tag{2.8}$$

Therefore, for $1 \leqslant r < e$, we shall have $\Omega[r] > r$ and the derived set $\{u_{m,n}(z;w)\}$ will not be effective in the spheres $\bar{S}_r$, where the given set $\{p_{m,n}(z;w)\}$ is effective.

3. Derived Sets

Before proceeding to prove Theorem 1, we must first carry out a study on the derived sets $\{u_{m,n}(z;w)\}$, of the given simple monic set $\{p_{m,n}(z;w)\}$, as defined by (1.8). In fact, introducing (1.1) in (1.8) we easily obtain

$$u_{m,n}(z;w) = \sum_{k=0}^{m+n}\sum_{j=0}^{k} (k+1-j)\, p^{m+1,n}_{k+1-j,j}\, z^{k-j}w^j, \tag{3.1}$$

noting that $p^{m+1,n}_{m+1,n} = 1$ and $p^{m+1,n}_{m+n+1-j,j} = 0$ for $j > n$. We also note that, although the set $\{u_{m,n}(z;w)\}$ is simple, yet the derived polynomials $(\partial/\partial z)\, p_{0,n}(z;w)$, for $n \geqslant 1$, are redundant and they must be eliminated. For this aim we write

$$\frac{\partial}{\partial z} p_{0,n+1}(z;w) = q_n(z;w) \qquad (n \geqslant 0), \tag{3.2}$$

and observing that the degree of $q_n(z;w)$ does not exceed $\frac{1}{2}n(n+1)+n$ (corresponding to the monomial w^n), we suppose that

$$q_n(z,w) = \sum_{h=0}^{n}\sum_{j=0}^{k} \alpha^n_{k-j,j}\, u_{k-j,j}(z;w) \qquad (n \geqslant 0). \tag{3.3}$$

To obtain recurrence relations for the coefficients $(\alpha^n_{h,i})$, we differentiate the representation (1.2) for w^{n+1} and then apply the relations (1.8), (3.2), and

(3.3). Noting that the simple set $\{p_{m,n}(z; w)\}$ is monic, we can easily arrive at the following relations.

$$\alpha^n_{n-j,j} = -\bar{p}^{0,n+1}_{n+1-j,j} \qquad (0 \leqslant j \leqslant n), \tag{3.4}$$

$$\alpha^n_{k-j,j} = -\bar{p}^{0,n+1}_{k+1-j,j} - \sum_{h=k}^{n-1} \bar{p}^{0,n+1}_{0,h+1} \alpha^h_{k-j,j} \qquad (0 \leqslant j \leqslant k; 0 \leqslant k \leqslant n-1). \tag{3.5}$$

It remains now to derive the $z^m w^n$-representation in terms of the polynomials $\{u_{m,n}(z; w)\}$ by differentiation of the expression (1.2) for $z^{m+1}w^n$, applying the definition (1.8) and (3.2) and then inserting the relation (3.3). In this way, we obtain the representation

$$z^m w^n = \frac{1}{m+1} \left[\sum_{k=0}^{m+n} \sum_{j=0}^{k} \bar{p}^{m+1,n}_{k+1-j,j} u_{k-j,j}(z; w) + \sum_{k=0}^{m+n-1} \sum_{j=0}^{k} \sum_{h=k}^{m+n-1} \bar{p}^{m+1,n}_{0,h+1} \alpha^h_{k-j,j} u_{k-j,j}(z, w) \right], \tag{3.6}$$

noting again that $\bar{p}^{m+1,n}_{m+1,n} = 1$ and $\bar{p}^{m+1,n}_{m+n+1-j,j} = 0$ for $j > n$. With the above study on derived sets we start to prove Theorem 1 in what follows.

Proof of Theorem 1

We first evaluate the coefficients $(\alpha^n_{\lambda,i})$ from the relations (3.4) and (3.5). In fact, in (3.5) we write j for $k-j$ and k for j; then in the notation (1.9), we obtain

$$\sum_{h=j+k}^{n} t_{n,h} \alpha^h_{j,k} = -\bar{p}^{0,n+1}_{j+1,k} \qquad (n \geqslant j+k). \tag{3.7}$$

For any fixed values of j and k, we consider the system of linear equations (3.7) in the unknowns $(\alpha^n_{h,k})$ for $n = j+k, j+k+1, j+k+2, \ldots$. It is easily seen, from the elementary theory of matrices, that the solution of the system (3.7) can be written in the form

$$\alpha^n_{j,k} = -\sum_{h=j+k}^{n} \bar{t}_{n,h} \bar{p}^{0,h+1}_{j+1,k} \qquad (n \geqslant j+k), \tag{3.8}$$

where the coefficients $(\bar{t}_{n,h})$ are introduced in the representation (1.10).

Now, since the set $\{t_n(z)\}$ is monic, then from (1.11) and (3.8), we can deduce that

$$|\alpha^n_{j,k}| \leqslant \lambda_n(r) \sum_{h=j+k}^{n} \frac{|\bar{p}^{0,h+1}_{j+1,k}|}{r^h} \qquad (n \geqslant j+k). \tag{3.9}$$

Furthermore, appealing to the relation (3.1) and applying Cauchy's inequality

$$|p^{j+1,k}_{h+1-i,i}| \leqslant M[p_{j+1,k}; r] \cdot \frac{\sigma_{h+1-i,i}}{r^{h+1}}$$

and noting that

$$(h+1-i)\frac{\sigma_{h+1-i,i}}{\sigma_{h-i,i}} < (h+1)e^{1/2},$$

we obtain

$$\begin{aligned} M[u_{j,k}; r] &\leqslant \sum_{h=0}^{j+k} \sum_{i=0}^{h} (h+1-i) |p^{j+1,k}_{h+1-i,i}| \cdot \frac{r^h}{\sigma_{h-i,i}} \\ &< \frac{e^{1/2}}{r} M[p_{j+1,k}; r] \sum_{h=0}^{j+k} \sum_{i=0}^{h} (h+1) \\ &< \frac{e^{1/2}}{3r}(j+k+1)(j+k+2)^2 M[p_{j+1,k}; r]. \end{aligned} \tag{3.10}$$

It should be observed that this inequality is true whether the given set $\{p_{m,n}(z; w)\}$ is monic or not.

We now evaluate the Cannon sum $\Omega_{m,n}[r]$ of the derived set $\{u_{m,n}(z; w)\}$ by appeal to the formula (3.6). Observing that

$$r^{h+1} < M[p_{0,h+1}; r],$$

since the set $\{p_{m,n}(z; w)\}$ is monic, and that

$$\frac{\sigma_{m,n}}{\sigma_{m+1,n}} < e^{1/2},$$

then, in view of (1.3), (3.9), and (3.10), we can easily arrive at the inequality

$$\begin{aligned} \Omega_{m,n}[r] &< \frac{e(m+n+1)(m+n+2)^2}{3r(m+1)} \omega_{m+1,n}[r] \\ &\times \left\{1 + \sum_{k=0}^{m+n-1} (k+1) \sum_{h=k}^{m+n-1} \frac{\lambda_h(r)}{r^h} \sum_{i=k}^{h} \frac{\omega_{0,i+1}[r]}{r^{i+1}}\right\}. \end{aligned} \tag{3.11}$$

We are now in a position to establish the inequality (1.13). If at least one of the functions $\lambda(r)$ and $\omega[r]$ is infinite there is nothing to prove. Supposing therefore that both the functions $\lambda(r)$ and $\omega[r]$ are finite, we choose the finite numbers τ and ρ to be respectively greater than $\lambda(r)$ and $\omega[r]$. Then from the definition (1.6) of $\omega[r]$ and (1.12) of $\lambda(r)$ it follows that

$$\omega_{m,n}[r] < K\rho^{m+n}; \qquad \lambda_n(r) < K\tau^n \qquad (m, n \geqslant 0), \tag{3.12}$$

where K denotes finite positive numbers independent of m, n and do not retain the same values at different occurrences. Introducing (3.12) in (3.11) and noting that $\tau > r$, $\rho > r$, it easily follows that

$$\Omega_{m,n}[r] < K(m+n)(m+n+1)^2(m+n+2)^2\, \tau^{m+n}(\rho/r)^{2(m+n)}, \tag{3.13}$$

for sufficiently large $m+n$. Making $m+n$ tend to infinity, (3.13) yields, for the Cannon function $\Omega[r]$,

$$\Omega[r] \leqslant \tau(\rho/r)^2,$$

and the required inequality (1.13) follows at once, in view of the choice of the numbers ρ and τ. The first assertion of Theorem 1 is now proved.

To complete the proof of Theorem 1 we assume that the given set $\{p_{m,n}(z; w)\}$ is effective in the closed sphere $\bar{S}_r$ so that $\omega[r] = r$ and the inequality (1.13) reduces to the form

$$\Omega[r] \leqslant \lambda(r). \tag{3.14}$$

The fact that this inequality cannot be improved is illustrated by showing that the bound $\lambda(r)$ is attained by the Cannon function of the derived set of the set $\{p_{m,n}(z; w)\}$ of the example of Section 2 above.

Actually, in the notation (1.9), we can deduce from the representation (2.4), written for w^{n+1}, that

$$t_0(z) = 1; \qquad t_n(z) = z^n + f(n)\left\{1 - \frac{1}{f(n+1)}\right\} z^{n-1} \qquad (n \geqslant 1).$$

Hence, the following representation can be verified.

$$z^n = t_n(z) + \sum_{k=0}^{n-1} (-1)^{n-k} \frac{f(k+1)}{f(n+1)} \left[\prod_{j=k+2}^{n+1} \{f(j) - 1\} \right] t_k(z) \qquad (n \geqslant 1).$$

Therefore, the Cannon sum of the set $\{t_n(z)\}$ will be

$$\lambda_n(r) = r^n + 2 \sum_{k=0}^{n-1} \frac{f(k+1)}{f(n+1)} \left[\prod_{j=k+2}^{n+1} \{f(j) - 1\} \right] r^k \qquad (r > 0). \tag{3.15}$$

Introducing the value (2.2) of $f(j)$ in (3.15), we obtain, on the one hand,

$$\lambda_n(r) > \prod_{j=1}^{n} \left(\frac{j+3}{j+1}\right)^{(j+1)/2}. \tag{3.16}$$

On the other hand, since $((j+3)/(j+1))^{(j+1)/2} < e$ and monotonically increases with j for $j \geq 0$, then (3.15) yields, for $0 < r < e$,

$$\lambda_n(r) < r^n + 2 \sum_{k=0}^{n-1} \left[\prod_{j=k+2}^{n+1} \left(\frac{j+3}{j+1}\right)^{(j+1)/2} \right] r^k.$$
$$< (2n+1)e^n. \tag{3.17}$$

Therefore, for the Cannon function of the set $\{t_n(z)\}$, (3.16) and (3.17) imply that

$$\lambda(r) = e \qquad (0 < r < e). \tag{3.18}$$

In view of the fact that the set $\{p_{m,n}(z; w)\}$ is effective in $\bar{S}_r$, for $r \geq 1$, it follows from (2.8), (3.14), and (3.18) that

$$\Omega[r] = e \qquad (1 \leq r < e),$$

and the bound in (3.14) is attained. The proof of Theorem 1 is therefore complete.

It should be finally observed that, if in the given simple (not necessarily monic) set $\{p_{m,n}(z; w)\}$, we have

$$p_{0,n}(z; w) = w^n \qquad (n \geq 1), \tag{3.19}$$

then in the derived polynomials there are no redundant ones. Hence the coefficients $(\alpha^n_{h,i})$ of (3.3.) no longer exist and from (3.6), we deduce that

$$z^m w^n = \frac{1}{m+1} \sum_{k=0}^{m+n} \sum_{j=0}^{k} \bar{p}^{m+1,n}_{k+1-j,j} u_{k-j,j}(z; w), \tag{3.20}$$

with the reservation that $\bar{p}^{m+1,n}_{m+1,n} \neq 0$ and $\bar{p}^{m+1,n}_{m+n+1-j,j} = 0$ for $j > n$. We now introduce in (3.20) the inequality (3.10), which is true whether the given set $\{p_{m,n}(z; w)\}$ is monic or not, and follow a treatment similar to that leading to the inequality (1.13). In this way we are led to the following inequality for the Cannon function of the derived set $\{u_{m,n}(z; w)\}$,

$$\Omega[r] \leq \omega[r], \tag{3.21}$$

provided that the Cannon function $\omega[r]$, of the given set $\{p_{m,n}(z; w)\}$, is finite. The inequality (3.21), which is true whether the set $\{p_{m,n}(z; w)\}$ is monic or not, will be supplemented in the following section on integrated sets.

4. Integrated Sets

Proof of Theorem 2

The proof is rather straightforward; thus, when $j \geqslant 1$, we evaluate $M[v_{j,k}; r]$ from the definition (1.14) and by the use of Cauchy's inequality. The following steps are easily arrived at:

$$M[v_{j,k}; r] \leqslant \sum_{h=0}^{j+k-1} \sum_{i=0}^{h} \frac{|p_{k-i,i}^{j-1,k}|\, r^{h+1}}{(h-i+1)\,\sigma_{h-i+1,i}}$$

$$\leqslant rM[p_{j-1,k}; r] \sum_{h=0}^{j+k-1} \sum_{i=0}^{h} \frac{\sigma_{h-i,i}}{(h-i+1)\,\sigma_{h-i+1,i}}$$

$$< \frac{1}{2} e^{1/2} r(j+k)(j+k+1)\, M[p_{j-1,k}; r] \tag{4.1}$$

Further, the representation of the monomials $(z^m w^n)$ in terms of the polynomials $\{v_{m,n}(z; w)\}$ can be derived from (1.2) in the form

$$z^m w^n = m \sum_{k=1}^{m+n} \sum_{j=0}^{k-1} \bar{p}_{k-1-j,j}^{m-1,n} v_{k-j,j}(z; w), \tag{4.2}$$

when $m > 0$, and when $m = 0$ then $w^n = v_{0,n}(z; w)$ as in (1.15).

The Cannon sum $\Lambda_{m,n}[r]$ of the integrated set $\{v_{m,n}(z; w)\}$ can be deduced from (4.1) and (4.2). Thus, when $m > 0$, the following inequality can be easily obtained.

$$\Lambda_{m,n}[r] < \tfrac{1}{2} er(m+n)^2(m+n+1)\, \omega_{m-1,n}[r], \tag{4.3}$$

and when $m = 0$, (1.15) gives

$$\Lambda_{0,n}[r] = r^n. \tag{4.4}$$

Now, if the Cannon function $\omega[r]$ of the set $\{p_{m,n}(z; w)\}$ is finite, then given a finite number ρ greater than $\omega[r]$, we can deduce from (4.3) and (4.4) that

$$\Lambda[r] \leqslant \rho,$$

where $\Lambda[r]$ is the Cannon function of the set $\{v_{m,n}(z; w)\}$. Hence by the choice of the number ρ we may infer that

$$\Lambda[r] \leqslant \omega[r]. \tag{4.5}$$

Finally, we observe that the given simple set $\{p_{m,n}(z; w)\}$ is the derived set of the set $\{v_{m,n}(z; w)\}$ in which we have

$$v_{0,n}(z, w) = w^n \qquad (n \geqslant 0),$$

so that Eq. (3.19) is satisfied. Therefore, if $\Lambda[r]$ is finite the inequality (3.21) is valid, and in the foregoing notation, it can be written in the form

$$\omega[r] \leqslant \Lambda[r]. \tag{4.6}$$

The inequalities (4.5) and (4.6) imply the equality of the Cannon functions $\Lambda[r]$ and $\omega[r]$ and Theorem 2 is the therefore established.

In conclusion, we append with the following result.

THEOREM 3. *Let $\{p_{m,n}(z; w)\}$ be a given simple set of polynomials which satisfy* (3.19). *Suppose further that*

$$p_{m,n}(0; w) = 0 \qquad (m \geqslant 1; n \geqslant 0), \tag{4.7}$$

and that $\{u_{m,n}(z; w)$ is the derived set of the given set $\{p_{m,n}(z; w)\}$. Then the Cannon functions of the sets $\{p_{m,n}(z; w)\}$ and $\{u_{m,n}(z; w)\}$ are equal for any positive value of r for which these functions are finite.

In fact, according to (1.8) and (4.7), we observe that

$$p_{m,n}(z; w) = \int_0^z u_{m-1,n}(\zeta; w)\, d\zeta \qquad (m \geqslant 1; n \geqslant 0).$$

Hence, the required result of the theorem follows from Theorem 2, in view of the Eq. (3.19).

Consideration of the set $\{p_{m,n}(z; w)\}$, given by

$$p_{m,n}(z; w) = \frac{(m+n)!}{\sigma_{m,n}} + z^m w^n \qquad (m \geqslant 1; n \geqslant 0),$$

$$p_{0,n}(z; w) = w^n \qquad (n \geqslant 0),$$

illustrates the essentiality of the condition (4.7).

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